MATHEMATICS

and COMMON SENSE

A Case of Creative Tension

Philip J. Davis

Mathematics and its applications are amphibians that live between common sense and the irrelevance of common sense, between what is intuitive and what is counterintuitive, between the obvious and the esoteric. The tension that exists between these pairs of opposites is a source of its creative strength.

Addressed to all who are curious about mathematics and who wonder about its nature and the role it plays in society, this book provides discussions and examples from the simple to the more abstruse.

Davis

MATHEMATICS *and* COMMON SENSE

A K PETERS

Mathematics and Common Sense

Mathematics and Common Sense

A Case of Creative Tension

Philip J. Davis

A K Peters, Ltd.
Wellesley, Massachusetts

Editorial, Sales, and Customer Service Office
A K Peters, Ltd.
888 Worcester Street, Suite 230
Wellesley, MA 02482
www.akpeters.com

Library of Congress Cataloging-in-Publication Data
Davis, Philip J., 1923-
Mathematics and common sense : a case of creative tension / Philip J. Davis.
p. cm.
Includes bibliographical references.
ISBN-13: 978-1-56881-270-0 (alk. paper)
ISBN-10: 1-56881-270-1 (alk. paper)
1. Logic, Symbolic and mathematical--Methodology. 2. Reasoning.
3. Common sense. 4. Knowledge, Theory of. I. Title.
QA9.2D38 2006
511.3--dc22

2006012622

Cover images:
First marigold: See pages 4 and 156
Second marigold: Photo by Matt Fitt, www.mattfitt.com
Bee: Photo by Paul F. Worsham

Set in Times New Roman PS MT and Bell Gothic

Printed in India

10 09 08 07 06 10 9 8 7 6 5 4 3 2 1

Contents

Preface

The impetus for putting together this book came from a number of questions about mathematics asked of me by a woman I shall call Christina. Christina had the usual mathematical education of students who were not studying science and technology. She married a mathematician and, over the years, developed an increasing desire to know something more about mathematics. She pointed out to me that what she learned from the media left her with an image of the subject as an arcane, esoteric discipline created by unworldly, mentally disturbed geniuses totally lacking in social skills. The media—newspapers, TV, plays, science fiction, etc.—all deal with what might be called the sensational aspect of the subject: famous problems solved after centuries of work, a new prime number, of almost eight million digits, discovered. She felt, and I agree with her, that these aspects of mathematics, while undoubtedly interesting, did not in any way describe the subject as it was pursued by the average professional mathematician, nor how what they did affected our lives.

Christina asked me a number of questions about mathematics. I answered them, and she was pleased to the point of sending me another batch. Her questions and my responses to them make up the first part of this book. The second part is an elaboration of them.

From the age of about ten, I have studied mathematics, learning its history, applying it, creating new mathematics, teaching it, and writing about it. This has left me with a personal and somewhat idiosyncratic view of the subject (or so my colleagues tell me). I would like to share a bit of this view with my readers.

Mathematics is a subject that is one of the finest, most profound intellectual creations of humans, a subject full of splendid architectures of thought. It is a subject that is also full of surprises and paradoxes. Mathematics is said to be nothing more than organized common sense, but the actuality is more complex. As I see it, mathematics and its applications live between common sense and the irrelevance of common sense, between what is possible and what is impossible, between what is intuitive and what is counterintuitive, between the obvious and the esoteric. The tension that exists between these pairs of opposites, between the elements of mathematics that are stable and those that are in flux, is a source of creative strength.

I go on to other paradoxes. Mathematics is a subject based on logic and yet formal logic is not a requirement for a higher degree. It is thought to be precise and objective with no whiff of subjectivity entering to compromise its purity, and yet it is full of ambiguities. It is the "language that nature speaks" but neither scientists nor philosophers have as yet provided cogent reasons for this.

Mathematics is also an attitude and a language that we employ by fiat and in increasing amounts to give order to our social, economic and political lives. It is a language, a method, an attitude that has diffused into medicine, cognitive science, war, entertainment, art, law, sports; which has created schools of philosophy, which has given support to views of cosmology, mysticism, and theology. All this adds up to a spectacular performance for a subject that in elementary grades has amounted to a rigid set of rules that say "do this and do that."

This book is addressed to all who are curious about the nature of mathematics and its role in society. It is neither a textbook nor a specialists' book. It consists of a number of loosely linked essays that may be read independently and for which I have tried to provide a leitmotif by throwing light on the relationship between mathematics and common sense. In these essays I hope to foster a critical attitude towards both the existence of common sense in mathematics and the ambiguous role that it can play.

These essays show how common sense bolsters or creates conflicts within the ideas and constructions of pure mathematics. Among other things, the reader will find discussions of the nature of logic and the uses of inconsistency, discussions of numbers and what to do with them, of conceptions of space, of mathematical intuition and creativity, of what constitutes mathematical proof or evidence. I will also discuss how common sense operates when mathematics together with its operations engages a wide variety of problems posed by the world of objects of people and events.

Writer Anne Fadiman tells how consternation arose when she tried to amalgamate her substantial collection of books with her husband's equally substantial collection. Each had a "common sense" system of filing; but the systems were mutually incompatible. Later, both Anne and her husband were in minor shock when they heard that after their friend's apartment had been redecorated, the decorator, exercising his own common sense, restored all the books to the shelves according to size and color.

There are more than sixty major subjects or branches of mathematics many of which have significant connections to the other branches. There is therefore no way that the mathematical corpus can be put into a linear order that is totally sensible and consistent. Though the present book alludes to only a very few of these subjects, its chapters may display a certain helter-skelter quality in their arrangement and a slight redundancy in their content. I ask for my readers' indulgence.

I hope that the readers of this book will come out with an appreciation of the flights of human imagination that both join and transcend common sense and that have created the mathematical world we live in.

The Further Reading sections in this book contain references to material that is both popular and professional.

Further Reading

Anne Fadiman. *Ex Libris: Confessions of a Common Reader*. Farrar, Straus, Giroux, 1998.

"Playing Ball Probabilistically." *SIAM News*, Vol. 35, No. 3, April 2002.

"Mumbo Math." *SIAM News*, Vol. 37, No. 9, November 2004.

"When Mathematics Says No." In *No Way: The Nature of the Impossible.* Philip J. Davis and David Park, eds., pp. 172 ff. W. H. Freeman, 1987.

"Paraconsistent Thoughts about Consistency." *Mathematical Intelligencer*, Vol. 24, No. 2, Spring 2002.

"Letter to Christine." *Newsletter, European Mathematical Society*, March 2004. Also in *Matilde: Nyhedsbrevfor Dansk Matematisk Forening*, Nummer 20, Juni 2004.

"Applied Mathematics as Social Contract." *Zentralblatt für Didaktik der Mathematik*, 88/I. Also in *The Mathematics Magazine*, Vol. 61, No. 3, June 1988, pp. 139–147.

"When Is a Problem Solved?" In *Current Issues in the Philosophy of Mathematics From the Viewpoint of Mathematicians and Teachers of Mathematics,* Bonnie Gold and Roger Simons, eds. Mathematical Association of America, 2006.

To

Christa and Peter

Christine and Fritz

and

Kay

in friendship

We are enlarged by what estranges.

—Richard Wilbur, "A Wall in the Woods: Cummington"

This is the vision ... to endure ambiguity in the movement of truth and to make light shine through it.

—Jonah Gerondi (c. 1200–1263), *The Gates of Repentance*

Letters to Christina: Answers to Frequently Asked Questions

The Swiss mathematician Leonhard Euler wrote a highly admired three-volume popularization of science entitled *Letters to a Princess of Germany* (published: 1768–1772). Some time ago, my friend Christina (name changed) sent me a letter with a number of questions about mathematics, a subject about which she had heard much but knew little. I believe that her questions are among those frequently asked by the general public.

Christina's questions and my answers provide an overview of the mathematical content of this book and set the stage for the essays that follow.

Q1. What is mathematics?

A1. Mathematics is the science, the art, and a language of quantity, space, and pattern. Its materials are organized into logically deductive and very often computational structures. Its ideas are abstracted, generalized, and applied to topics outside mathematics.

The mix of mathematics and outside topics is called *applied mathematics*. Mathematics has often been called the "handmaiden of the sciences," because of its use in and interactions with the other sciences. For example, for reasons that are by no means clear, mathematics is an indispensable aid to the physical sciences. The expositions of theoretical physics are completely mathematical in character.

Applied mathematics includes *descriptions, predictions,* and *prescriptions.* Description replaces a real-world phenomenon with a mathematical surrogate: A lampshade casts a parabolic shadow on the wall. Prediction makes a statement about future events: A total eclipse of the sun will occur on July 11, 2010; its duration will be 5 minutes and 20 seconds. Prescription (or formatting) organizes our lives and actions along certain lines: Traffic lights control the flow of automobiles in a periodic fashion; tax laws affect us all.

Mathematics has intimate relations with philosophy, the arts, language, and semiotics (the theory of signs and their use). Since the development of mathematics is often inspired and guided by aesthetic considerations, mathematics can be described as “amphibious”: It is both a science and one of the humanities.

Q2. Why is mathematics difficult, and why do I spontaneously react negatively when I hear the word?

A2. There are many reasons why the average person finds mathematics difficult. Some of them are poor, uninformed teaching; over-concentration on the deductive aspect of the subject; boring presentation; presentation that fails to connect mathematics with the day-to-day concerns of average people.

Mathematical thinking and manipulations are cerebral activities that, simply, not everyone enjoys. And then, let’s face it, the material can be difficult, common sense seems to be irrelevant. While mathematical skills and understanding can be learned and developed, I believe there is such a thing as innate talent for mathematics. Just as not everyone has the talent to create art, write a great book, be a ballet dancer, or break athletic records, not everyone can scale the heights of mathematical understanding. The very fact that professional mathematicians make long lists of unsolved problems attests to the fact that ultimately, all mathematicians reach their own limits of mathematical accomplishment.

Q3. Why should I learn mathematics? History widened my horizons and deepened my "roots." When I learned German, it opened up cultural treasures to me. Karl Marx explained (if not changed) the society I live in. What does mathematics have to offer?

A3. I’ll start by providing the time-honored reason. Mathematics trains us to think logically and deductively. I would rephrase this by saying that mathematics opens up the *possibility* of rational thought in contrast to unreasoned or irrational, or mystical, thought.

Mathematics is one of the greatest human intellectual accomplishments. We learn mathematics partly to enable us to function in a complex world with some intelligence, and partly to train our minds to be receptive to intellectual ideas and concepts. Even as natural languages such as German or Chinese embody cultural treasures, so does mathematics: One has to learn the "language" of mathematics to appreciate its treasures. To learn the history of mathematics is to appreciate its growth within the history of general ideas.

Looking about us, if we are perceptive, we can see not only the natural world of rocks, trees, and animals, but also the world of human artifacts and human ideas. The ideas of Karl Marx (and of other thinkers) explain the world along certain lines, each necessarily limited by the thinker's own constraints. Mathematics explains the world in remarkable but limited ways. It is capable of formatting our lives in useful ways, for example, when we take a number for our "next" at the deli, and of allowing us to look into the future and make prudential judgments; for example, the meteorological predictions based on mathematics theories. Every educated person should achieve some appreciation of the historical role that mathematics plays in civilization in order to give the subject both intelligent support and—when it seems necessary—intelligent resistance to its products.

Q4. How has mathematics changed in the last 100 years? What have been the dominant trends?

A4. The number of new mathematical theories produced since 1900 is enormous. During that time, mathematics has become more abstract and deeper than it was in the 19th century. By this I do not mean that simple things such as arithmetic have to be viewed by the general public in a more abstract or deeper way than before, but that the conceptualization of old mathematics and the creation and applications of new mathematics by mathematicians have had that character.

Mathematical logic is now firmly on the scene, as is set theory. A new and more abstract algebra has grown mightily and consolidated itself. Infinitary mathematics (i.e., the calculus and its elaborations) has grown by leaps and bounds, but recently has had increasingly to share the stage with finitary mathematics.

Geometry, which began in antiquity as visual and numerical (lengths, areas, volumes), moved into an abstract axiomatic/deductive mode and now includes such topics as algebraic, combinatorial, and probabilistic geometry.

A major trend since the late 1930s is to view mathematics as the study of deductive structures. Mathematics has shared a structuralist point of view with numerous non-mathematical disciplines ranging from linguistics to anthropology to literary criticism.

There is no doubt that the electronic digital computer that emerged in the middle 1940s, which is in many ways a mathematical instrument, has changed our day-to-day life noticeably. The computer has changed mathematical education. It has revitalized or widened the scope and increased the power of a number of traditional branches of mathematics. It has also created new branches of the subject and, all in all, has had a revolutionary effect throughout science and technology.

Q5. What can you tell me about the "chipification" of mathematics?

A5. A large part of the mathematics that affects our day-to-day lives is now performed automatically. Computer chips are built into our wristwatches, automobiles, cash registers, ATMs, coffee machines, medical equipment, civil-engineering equipment, word processors, electronic games, telephone and media equipment, military equipment, IDs... The list is endless, and the idea of inserting chips to do clever things has become commonplace.

The mathematics in these chips is hidden from view. The average person, though greatly affected by it, needs to pay no conscious attention to its mathematics. That is the job of the corps of experts who design, implement, monitor, repair, and improve such systems. The "chipification of the world" is going forward at a dizzying rate. One of the disturbing side effects of this trend may very well be that in the near future our sons and daughters will not learn to add or read in the conventional way. Will this put us back to several hundred years ago, when literacy and numeracy were relatively rare achievements? Not at all. Modes of communication, interpretation, and social arrangements will change; it does not mean that they will disappear.

Q6. Where are the centers that develop new mathematics? Who is "hip"? A fan of good rock music knows where to go to hear it. Where would a math fan go these days?

A6. There are mathematical centers in all the developed countries of the world. They are located in universities; governmental agencies; research hospitals; scientific, economic or social "think tanks"; and industries. All these centers tend to be organized into groups and tend to concentrate their research on very specific problems or subject matter. There are also many mathematical researchers who work independently and productively outside of groups.

Depending on what sort of "mathematical music" you would like to hear, you would visit one of these groups or individuals. Although the groups exhibit a great deal of *esprit de corps* and self-esteem and can be influential beyond their own walls, there is, in my judgment, no single group that is predominates; this diversity is a very good thing. Closer to home, there are now websites that cater to every level of mathematical interest and engagement.

Q7. How is mathematics research organized? Who is doing it, who is paying for it, and why? Lonely, harmless "riders" or highly efficient, highly organized, secret, threatening groups? Should one be scared?

A7. The forward movement of mathematics is driven by two principal forces: forces on the inside of the subject and forces on the outside.

Forces on the inside perceive certain questions or aspects that are incomplete, unanswered, and call for answers. New mathematical ideas arising from free mental play can come into prominence. "Lonely riders" very often make contributions.

Forces on the outside call for the application of available mathematics to the outside world of people or things. These applications may result in the development of genuinely new mathematics.

The work is paid for by institutions (including universities) or organizations in anticipation of profit, or of scientific, social, national, or cultural gain. The economically independent mathematical researcher exists but is quite rare. The space agency NASA is, for example, one source of considerable mathematical support.

In the years since 1939, and particularly in the United States, a great deal of mathematical research has been paid for by the military. Individual mathematicians have profited from this support independent of their personal views of the United States military or its foreign policy. Since war appears to be endemic to the human situation, this support will undoubtedly continue.

Other support comes from the medical and healthcare sector, and this support is likely to increase in importance. The productivity of the United States gets a considerable boost from the entertainment industry, and today's movies use computer graphics software with a considerable mathematical underlay.

Some of the work of the various groups is restricted or confidential. This may be for reasons of national security or for reasons of industrial confidentiality in a free and competitive market.

In an ideal democratic society, all work must ultimately come into the open so that it may be judged both for its internal operation and its effects on society. Free and complete availability of information is one of the hallmarks of an ideal science. While this has not always been the case in practice, judging from the past several centuries, the record is pretty good. Doing mathematics, developing new mathematics, is simply one type of human activity among thousands of activities. As long as constant scrutiny, judgment, and dissent flourish, fear can be reduced.

Q8. Other sciences have had breakthroughs in the 1980s and 1990s. What breakthroughs does mathematics claim?

A8. We might begin by discussing just what is meant by the phrase "a breakthrough in mathematics." If we equate breakthroughs with prizes, and this equivalence has a certain merit, then the work of, e.g., the Fields Medal winners should be cited. This would cover pure mathematics, where this prize, given since 1936, is often said to be the mathematical equivalent of the Nobel Prize. Similar prizes of substantial value exist in various pure mathematical specialties, applied mathematics, statistics, computer science, etc.

The work of Fields Medal winners over the past decades, people such as K. F. Roth, R. Thom, M. Atiyah, P. J. Cohen, A. Grothendieck,

S. Smale, A. Baker, K. Hironaka, S. P. Novikoff, and J. Thompson, cannot, with some very few exceptions, be easily explained in lay terms. In view of their arcane nature, the public rarely hears about such things. Should a particular accomplishment reach the front page of the newspapers, its meaning is often mutilated by packaging it in the silver paper of sensation. Sensation is always easier for the public to grasp than substance.

Judging from the selection of Fields Laureates, the criteria for the honor seem to be (1) the solution of old and difficult mathematical problems, (2) the unification of several mathematical fields through the discovery of cross connections and of new conceptualizations, and/or (3) new internal developments.

A few years ago the world of research mathematicians was electrified by the public announcement that the Clay Mathematics Institute, a private organization, was offering $1 million each for the solutions of seven famous mathematical problems.

We live in an age of sensation. As a result, only the sensational aspects of mathematics get much space in the newspapers: the solution to Fermat's "Last Theorem," "the greatest prime number now known is...," the solutions of other so-called "big" unsolved problems, international students' contests, etc.

Mathematical applications that are useful, e.g., the programming and chipification of medical diagnostic equipment, are rarely considered to be breakthroughs and make the front pages. Historic research or philosophic discussions of the influence of mathematics on society have yet to be honored in prestigious manners.

From the point of view of societal impact, the major mathematical breakthrough since the end of World War II is the digital computer in all its ramifications. This breakthrough involved a combination of mathematics and electronic technology and is not the brainchild of a single person; hundreds, if not hundred of thousands, of people contributed and still contribute to it.

Q9. Medical doctors fight cancer, AIDS, and SARS. What is now the greatest challenge to modern mathematics?

A9. One might distinguish between "internal" problems and "external" problems. The former are problems that are suggested by the operation of the mathematical disciplines themselves. The latter are problems that come to it from outside applications, e.g., what airplane shape has the minimum drag when the airplane surface is subject to certain geometrical conditions? What are the aerodynamic loads on the plane structure during maneuvers?

In 1900, the mathematician David Hilbert proposed a number of very significant unsolved problems internal to mathematics. Hilbert's reputation and influence was so great that these problems have been worked on steadily, and most of them have been solved. Setting up, as it did, a hierarchy of values as to what was important (every mathematician creates his own list of unsolved problems!), this list has had a considerable influence on the subsequent progress of mathematics. The solvers, in turn, have gained reputations for themselves in the mathematical community.

Within any specific field of mathematics, the practitioners will gladly tell you what they think the major unsolved problem (or challenge) is. Thus, if a topologist is queried, the answer probably is to prove the Poincaré Conjecture in the case $n = 3$. If an analyst is queried, the answer might be "Prove the Riemann Hypothesis."

As for external problems, a fluid dynamicist might say "Devise satisfactory numerical methods for processes [such as occur in turbulence or in meteorology] that develop over long periods of time." A programming theorist might say "Devise a satisfactory theory and economic practice for parallel computation."

If your question is answered in terms of specific problems, it is clear that there are many of them, and there is no agreement on how to prioritize them. Researchers can be drawn to specific problems by the desire for fame, or money, or simply because their past work suggests fruitful approaches to unsolved problems.

Your question can also be answered at a higher level of generality. A pilot assessment of the mathematical sciences prepared for the United States House (of Representatives) Committee on Science, Space, and Technology, identifies five interrelated long-term goals for the mathematical sciences. These are

- to provide fundamental conceptual and computational tools for science and technology;
- to improve mathematics education;

- to discover and develop new mathematics;
- to facilitate technology transfer and modeling;
- to promote effective use of computers.

[*Notices of the American Mathematical Society*, February 1992]

I would like to go up one more rung on the ladder of generality and answer that the greatest challenge to modern mathematics is to keep demonstrating to society that it merits society's continued support. The long history of mathematics exhibits a variety of mathematical intents. Some of these have been to discover the key to the universe, to discover God's will (thought to be formulated through mathematics), to act as a "handmaiden" to science, to act as a "handmaiden" to commerce and trade, to provide for the defense of the realm, to provide social formats of convenience and comfort, to develop a super brain—an intelligence amplifier of macro proportions.

Mathematics can and has flourished as a harmless amusement for a few happy aficionados both at the amateur and professional levels. But to have a long and significant run, mathematics must demonstrate an intent that engages the public. If the intent is simply to work out more and more private themes and variations of increasing complexity and of increasing unintelligibility to the general public, then its support will be withdrawn.

The public demands something in return for its support, but the place and the form of an acceptable return cannot be specified in advance. Perhaps a mathematical model of brain operation will be devised and will lead to insight and ultimately to the alleviation of mental disease. It is not too fatuous to think that many of the common problems that beset humankind such as AIDS, cancer, hunger, hostility, and envy, might ultimately be aided by mathematical methods and computation. But while efforts in these directions are praiseworthy, they may come to naught.

By way of summary, the greatest challenges to mathematicians are to keep the subject relevant and to make sure that its applications promote human values.

Q10. What can you say about the militarization, centralization, regionalization, and politicization of mathematics?

A10. This is a very wide-ranging question: Militarization alone would require several books. Since about 1940, one of the major financial supporters of mathematical research in the United States has been the military or the “military industrial complex”; a similar statement can be made of all the advanced nations. There are essential mathematical underlays to new, sophisticated weaponry, both offensive and defensive, and to military information processing systems. Related economic, demographic, strategic studies and predictions often involve complex mathematics.

Since the end of the Cold War, with the development of a variety of insurgencies and terrorist strategies, military options have been and are being reassessed, which will undoubtedly lead to new developments in mathematics. We can also foresee a time when the development and application of mathematics will be increasingly supported by fields such as medicine, biology, environment, transportation, finance, etc.

Centralization and regionalization: In the pre-computer days it was said that a mathematician didn’t require much in the way of equipment: a few reams of paper, a blackboard, and some penetrating ideas. Today, although much, and perhaps most, research is still done that way, we find increasingly that mathematicians, particularly applied mathematicians, require the aid of supercomputers. This is still far less costly than the billions of dollars of laboratory equipment required by high-energy physicists or astrophysicists.

Centralization of mathematics occurs as the result of a number of factors, including the willingness and ability of a society to support mathematical activity and the desire to have mathematicians work in groups or centers. The notion of a critical intercommunicating mass of creative individuals is at work here. New systems of rapid intercommunication and the transport of graphical and printed material may affect the clumping of future centers for research and development.

Politicization: While mathematical content is abstract, mathematics is created by people and is often applied by people to people. It is to be expected, then, that the creation and application of mathematics should be subject to support, pressure, monitoring, and suppression by governmental, political, or even religious institutions. The interaction between mathematics and human institutions has a long and documented history.

Q11. Give me ten points that worry a concerned mathematician.

A11. A concerned mathematician will worry about the abuse, misuse, or misinterpretation of mathematics or its applications. Insofar as we are living in a thoroughly mathematized civilization, the number of concerns is necessarily vast. Many such concerns focus on "life and death" issues. If, for example, a mathematical criterion were developed via encephalography for determining when a person is brain dead, then this would engender a great deal of concern.

Q12. I read a statement attributed to the famous physicist Max Born that the destructive potential of mathematics is an imminent trend. If that is so, why should I, an average person, learn more about mathematics?

A12. You should learn more about it for precisely that reason.

All creative acts have destructive potential. To live is to be at risk, and no amount of insurance can reduce the risk to zero. Moreover, to live at the very edge of risk is thought by some people to make them feel "truly alive." Electrical outlets in the home are not totally risk-free; the destructive or revolutionary potential of graphical arts, or of literature, is well documented. Even as mathematics solves many problems, it creates new problems, both internal and external.

The more the average person knows about mathematics, the better off that person is to make judgments. Some of those judgments will be about how to temper risk with prudence. In a world in which scientific, technological and social changes occur rapidly, a democratic society cannot long endure in the presence of ignorance.

Q13. What is deep mathematics and what is not?

A13. A quick answer is the one given by logician Hao Wang: A deep theorem is a formula that is short but can be established only by long proofs.

I, however, am going to answer differently than Hao Wang by changing your question just a bit and discussing the possibility that Mathematics with a capital M differs from mathematics with a lowercase m. First, a cautionary quotation:

> Insecure intellectuals make a false and basically harmful distinction between "high" and vernacular culture, and then face enormous trouble in trying to determine a status for significant items in between, like Gershwin's *Porgy and Bess* or the best of popular science writings.
>
> —*Crossing Over: Where Art and Science Meet,* Stephen J. Gould and Rosamond Wolff Purcell

Some months ago, I worked out a certain piece of mathematics that gave me much pleasure and that I believed was new and interesting. I thought about building it into a paper and then began to think to what periodical I might appropriately send it. Then I stopped short and said to myself: "You know, there is Mathematics with a big M and mathematics with a small m. What I've done here is of the small m variety. If I send it to periodical *XYZ*, it would be rejected out of hand."

What do I mean by Mathematics with a big M and mathematics with a small m? It would be impossible for me to present a list of criteria, and my criteria would not necessarily be my colleagues' criteria, but as the saying goes: "I know them when I see them." You have most likely heard of Art with a capital A and art with a small a. Possibly you've heard of Opera with a big O (grand Opera: Wagner, Verdi) and opera with a small o (opéra comique: Offenbach, light opera, Broadway musicals). Then there is poetry, verse, light verse, and doggerel. Recent articles have dealt with the movements back and forth within the categories of big and small. The work of Norman Rockwell—a popular American magazine cover artist, once thought to be the Rolls Royce of kitsch—has now been reconsidered and elevated in the minds of art critics.

I will end by adapting a paragraph of the philosopher William James. I have changed a few of James' words so the paragraph relates not to the split of philosophers between the tough- and tender-minded, but to mathematicians.

> It suffices for our immediate purpose that the M and m kinds of material, both exist. Each of you probably knows some well-marked example of each type, and you know what the authors of each type think of the authors on the other side of the line. They have low opinions of each other. Their antagonism, whenever as individuals their temperaments have been intense, has formed in all ages a part of the mathematical atmosphere of the time. It forms part of the mathematical atmosphere today.

Despite all the fuzziness and inconsistency (often in my own mind), the dichotomy remains alive in the mathematical world. It can be the source of professional snobbism: “X is not a ‘real’ mathematician,” “Y doesn’t prove theorems,” “Z only computes.” The split infects the way professional talks are given. It can play a role in job offers, promotions, and in obtaining contracts and grants. Though attitudes change, the dichotomy is not likely to go away soon.

1 What Is Mathematics?

As an elaboration of Q1–Q5 of "Letters to Christina," it is time to say a few more words about what mathematics is. Strangely, this question is hardly discussed in any course of mathematics routinely taken by undergraduates or graduates. In classes, mathematics is simply what the teacher slaps on the blackboard or displays via PowerPoint.

For starters, we might say that mathematics is the science of quantity and space. This answer might have been satisfactory four hundred years ago, but today we would say that mathematics is multi-faceted: It is the art and science of dealing with deductive (i.e., "theorematic") and algorithmic (i.e., computational) structures that concern themselves with quantity, space, pattern, and arrangement. Mathematics also deals with the language-like symbolisms that allow us to express and manipulate these concepts. It may be noted that these concepts and manipulations have changed over time, often growing, occasionally discarded.

While this definition is more in keeping with the spirit of our times than the brief "quantity and space," some specialists' interest groups—perhaps semioticians, logicians, applied mathematicians, physicists, non-Cantorian set theorists, and others—may feel that what I have just said overemphasizes certain aspects of mathematics at the expense of their own special concerns.

Philosophers of mathematics have also provided definitions of mathematics. Naturally they go for the aspects that are of philosophical interest: epistemology, ontology, theories of knowledge and cognition, semantics, semiotics, etc.

At a higher level of generality, Ruben Hersh and I have stated that "the study of mental objects with reproducible properties is called mathematics." It is a good thing that no definition of mathematics is legislated or comes down by ukase from a mathematical academy or from the dictionary makers. It is a good thing that, unlike doctors, there is no state licensing of mathematicians. This keeps mathematics fluid, as it should be. Mathematics is not in the hands of an academy, nor is it the exclusive property of the community of professional mathematicians. Mathematics lives and is shaped by all who contemplate, speculate on, describe, validate, apply, and develop mathematics.

It is not possible, however, to have mathematics be all things to all people. In any historical age, mathematics is and has to be what the age says it is. The dream of a *mathesis universalis* expressed by René Descartes and Gottfried von Leibniz among others, of an all-embracing mathematical formulation of the representations of the outer physical world, of the decisions, acts, and interpretations of the social world, is unrealistic.

René Descartes (1596–1650)

Despite a famous quotation to the effect that "the essence of mathematics lies entirely in its freedom," this should not be interpreted as saying that anything goes in mathematics. Descartes thought he was doing mathematics when he wrote the *Principles of Philosophy,* and for all I know Benedict Spinoza may have thought the same about his *Ethica More Geometrico Demonstrata,* which he composed *à la* Euclid with definitions, theorems and corollaries. I don't know whether anyone believed it at the time, but nobody would today.

A hundred advanced texts could not summarize the current state of mathematical knowledge. The *CRC Concise Encyclopedia of Mathematics* runs to almost 2000 pages and merely scratches the surface. A users' handbook for the software package *Mathematica* runs to almost 1500 pages, and computation is only a small part of mathematics.

As in any field, in mathematics there must be both expansion and contraction, or it will become closed and stagnate. Expansion exists in the unfolding of new constructs, new meanings, and new practices. Contraction exists in a variety of ways: as condensation or as the realization or declaration of irrelevance or self-limitation.

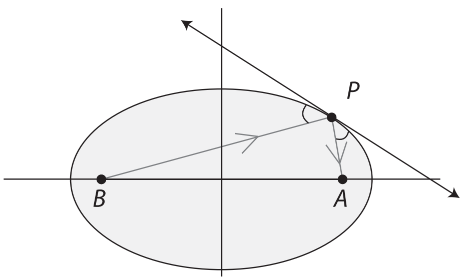
By way of illustrating various aspects of mathematics, below are a few examples of statements (or theorems) that display some of the often-overlapping elements of mathematics.

Quantity. $13 \times 24 = 312$; $13 \div 24 = 0.5416666...$

$$2 - \sqrt{2} = .61254732...$$

$$1 + (1/2^2) + (1/3^2) + (1/4^2) + \ldots = \pi^2/6$$

Space. A non-circular ellipse has two distinct foci; *A* and *B*. The sum of the distances from any point *P* on the ellipse to the foci is a constant.



The volume of a sphere of radius r is given by $4\pi r^3$.

A Möbius band is a one-sided surface.

There are only five regular polyhedra: the cube, the tetrahedron, the octahedron, the dodecahedron, and the icosahedron.

Patterns. The elements a, b, and c of a commutative group satisfy the algebraic patterns $(ab)c = a(bc)$ and $ab = ba$.

Pascal's triangle,

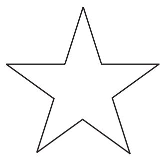
1
1 2 1
1 3 3 1
1 4 6 4 1
etc.

is a numerical pattern of integers in which a number in any row is the sum of the two numbers in the previous row and slightly to each side.

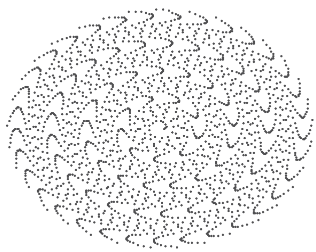
The numbers in Pascal's triangle are also known as the binomial coefficients and are found in a great many patterns.

Symmetry. One very important instance of geometric pattern is symmetry. An ellipse has two axes of symmetry.

The five-pointed star below has both rotational and reflective symmetries. If the star is rotated 72°, it will cover the original star exactly. If a line is drawn through the star from one point to the opposite side, and the star reflected over that line, the reflection will look the same as the original star.



The "marigold" figure below is certainly a geometric pattern. It even appears to have rotational symmetry. But this is not strictly the case.



There are logico-algebraic patterns such as

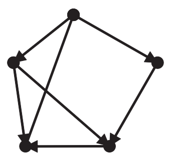
$$\varphi \wedge \psi \equiv \neg\varphi \downarrow \neg\psi$$

which is a symbolic representation of the statement "'both phi and psi' is logically equivalent to the statement 'neither (not phi) nor (not psi)'."

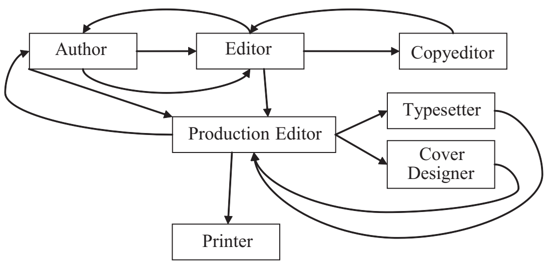
Arrangements. A deck of 52 cards can be arranged in $1 \times 2 \times 3 \times ... \times 52$, or 8.0658×10^{67}, ways. Theories of arrangement feed into computations of probability. The probability that a shuffle of 52 cards will produce aces as the top two cards is [4 aces]/[52 cards] × [3 aces]/[51 cards left] = $4/52 \times 3/51 = 1/13 \times 1/17$, or 0.004524....

Mathematical structures. A structure is a set of idealized objects that are related to one another in fixed ways or that combine through certain operations. Consider the set N of integers 1, 2, 3,.... Some rules for the combinations of integers are $a + b = b + a$, $ab = ba$, and $a(b + c) = ab + ac$. Within the set N, rules for carrying out the various operations of arithmetic, +, –, ×, /, √, are learned in elementary school and are built into digital computers.

A directed graph is a mathematical structure. What is important in the flow chart below are the nodes and links and not the individual designations given to the nodes.



A directed graph



A flow chart as a directed graph

Ruler and compass geometry, wherein points, lines, and circles are allowed to combine in certain specified ways constitutes a mathematical structure.

Mathematical structures carry an aura of exclusivity. They are like social clubs whose laws of admission are set up on the basis of "who shall we keep out?" Thus, the infinite sequence of ones (1, 1, 1,…) is excluded from membership in the Hilbert Space called H^2 because the sum of its squared components is infinite. The purpose of club formation is to insure that members interact properly. Within the club some things are possible for its members and other things impossible.

Deductive structures. The historic and paradigmatic example of this is Euclidean geometry as taught in high school or more sophisticated versions as developed in the late nineteenth century. Here certain "self-evident" statements (axioms) are accepted, from which certain geometric conclusions (theorems) are reached by way of certain accepted rules of logical procedure.

Language-like symbols or notation. Mathematical notations are often found embedded within sentences in natural languages. Here are two clips from technical papers:

"$\lim_{n\to\infty} [\varphi(n+x) - \varphi(n)] = 0$, for all $x \in (-1, \infty)$"

"Let $\psi(v)$ be a forcing arrangement with free variable v such that it's forced that for all $v\ \psi(v) \to v \in \omega$."

Figures. Mathematical discourse is often accompanied by graphical figures of various sorts, some of which are included above. Figures serve not only to clarify text and make it vivid but also to suggest new possibilities. Over the years, however, figures have played an ambiguous role, one that will be discussed at length in a later chapter.

Mathematics: Pure and Applied

Since definitions of mathematics are hardly ever discussed in mathematics classes, it's not surprising that even less discussion is directed toward the distinction between pure and applied mathematics. In elementary mathematical education the two types are intertwined: pure mathematics is manipulations in arithmetic and algebra; applied mathematics consists of "word problems." At higher levels, in college, the situation is somewhat different. A number of universities have two departments of mathematics: pure mathematics and applied mathematics, often at loggerheads.[1] Since the courses offered in these departments occasionally bear the same name, incoming freshmen are often confused as to which course they should take, or, if interested in majoring in mathematics, which type of mathematics they should take.

1. Cf. the old German double mathematical joke: there are two kinds of mathematics: *reine* (= *abgewandte*) *Mathematik* und *unreine* (= *angewandte*), i.e., pure and impure.

One could describe the difference between pure and applied mathematics as follows: Pure mathematics is inward looking while applied mathematics is outward looking. That is to say, pure mathematics looks to the study and development of its own theories and interpretations while applied mathematics stresses the application of mathematical theories to the physical and social worlds.

An example from pure mathematics is the theorem of Niels Henrik Abel that the roots of a polynomial of degree five or higher cannot, as a rule, be expressed in terms of integers subjected to the operations of elementary arithmetic and simple root extraction. This theorem (or fact) is of little concern to an applied mathematician.

In contrast, applied mathematics is close to physics, engineering, technology, and economics. Since these subjects are developed via mathematical language, there may be little distinction between them and applied mathematics. Applied mathematics, physics, econometrics, psychometrics, jurimetrics, and technology develop a large variety of mathematical theories or models for dealing with the subjects. If an applied mathematician comes up with a numerical scheme (an algorithm) for modeling the movements of a tornado, it may be of little interest to the pure mathematician. Then again, it may suggest some problems for the pure mathematician to tackle.

Another contrast: Pure mathematics is often pursued with an "art for art's sake" motivation or to escape from the bitter and seemingly unsolvable problems posed by life. Applied mathematics has a utilitarian (in the humanistic sense) motivation. Nonetheless, each type of mathematical concern draws on and contributes to the other, and the boundaries between the two fields are both fuzzy and flexible. The step from the "fantasies" of pure mathematics to real applications in engineering, in the sciences, in social organizations, can be small. Since the pursuit of both the pure and the applied requires financial support, pure mathematics approaches the public for financial support with the reasonable claim that its products are "potentially" applicable. It should not, then, come as a surprise that the relationships between common sense and these two "varieties" of mathematical activity exhibit rather different features and call into play different aspects of common sense.

Further Reading

Amy Dahan Dalmedico. "An Image Conflict in Mathematics after 1945." In *Changing Images in Mathematics: From the French Revolution to the New Millennium,* Umberto Bottazzini and Amy Dahan Dalmedico, eds., pp. 223–254. Routledge, 2001.

Philip J. Davis and Reuben Hersh. *The Mathematical Experience*. Birkhäuser Boston, 1980.

Philip J. Davis and Reuben Hersh. "Rhetoric and Mathematics." In *The Rhetoric of the Human Sciences*, John S. Nelson et al., eds. University of Wisconsin Press, 1987.

J. Hoffman, C. Johnson, and A. Logg. *Dreams of Calculus*. Perspectives on Mathematics Education, Springer, 2004.

Didier Norden. *Les mathématiques pures n'existent pas*. Paris: Actes Sud, 1981.

David Park. *The Grand Contraption: The World as Myth, Number and Chance.* Princeton University Press, 2005.

33 Mathematics at the Razor's Edge

You taught me language, and my profit on it is I know how to curse.
—Caliban, in William Shakespeare's *The Tempest*

I come finally to what is probably the most important and perplexing question relating to mathematics and common sense. Where is mathematics going, and what is the common sense of society's support of mathematical research and of its applications? What is the common sense—or the opposite—of creating more mathematics? Isn't there more than enough of it already?

Here are some answers that have been given over the centuries for why people work with mathematics, teach it, apply it, create more of it, philosophize about it, and even despise it. In each period of time, society has decided for which of these reasons it wishes to support mathematics with understanding, respect, honors, and with cold hard cash.

Technological. Mathematics is the language of science and technology. It helps create new physical theories. It helps predict the future. A good part of contemporary technology has a substantial mathematical base and carries with it the possibility of unintentional effects: A search engine is a wonderful device with many fine uses and now there is psychological evidence that it has reduced individual autonomy by overreliance on easily available information.

Economic. Mathematics is the binding language of economics and a facilitator of business and commerce.

Educational. Mathematics is one of the classical liberal arts. By stressing logical thinking, it helps develop skills in diverse areas; it enables us to live intelligently and productively in an increasingly mathematized world.

Societal. Mathematics can create techniques that can lead to the alleviation of the ills of society, and, as already noted by Archimedes (c. 225 BC) and Niccolo Tartaglia (1499–1557), it can facilitate the destruction of lives.

Medical. Mathematics can lead to the understanding of the human body and assist medical procedures. It can lead to the alleviation of pain and prolong human lives.

Personal. Mathematics is an avenue for intellectual competition. The possibility of engaging in it is there. It is fun to do and leads to aesthetic pleasure. It provides an escape from an imperfect world.

Philosophical. Mathematics is the essence of rational and objective thought and hence promotes such thought. It is worth doing for its own sake and promotes an art-for-art's-sake attitude to the subject. Over the years, mathematics has contributed mightily to philosophical concepts and positions. It leads to a deep knowledge of the cosmos and, ultimately, to a knowledge of God.

Mystical. Mathematics provides a mystical and occult interpretation of aspects of the universe.

***

Looking to the future, some of the immediate goals of mathematical research, described in general language, are to discover (or create) new mathematical concepts, theories, and relationships; to formulate new conjectures; to prove old and new conjectures. The specifics of mathematical programs for the future have been laid out in numerous publications, particularly at the time of the millennium (the years 2000–2001).

The experience of the past 500 years is that these goals have been fulfilled in ample measure, and, as to the future, mathematical physicists John C. Baez and James Dolan expect "the amount of mathematics produced in the 21st century to dwarf that of all the centuries that came before."

Digital computers and their interlinked networks, which are mathematical and informational engines and much more, have ushered in an era of intensified activity aimed at the mechanization of mathematics. As a corollary, some seers have said that in virtue of the tremendous facility of computer-generated graphics, mathematics will depend increasingly on visualizations and less on formulaic material. The old guard will then complain, as it has on several occasions in the past, that "this stuff isn't really mathematics," and the Young Turks will put forward a reciprocal opinion.

If mathematics is a totally formalizable discipline—a proposition that for numerous reasons I do not believe—then it can be turned over completely to computers. The computers will presumably then be sufficiently intelligent to discover—without the intervention of humans—deep mathematical phenomena. One might say then, paradoxically, that one of the successful aims of mathematical thought has been to get rid of mathematical thought.

If one interprets the word "mechanization" in a historical sense, this is a process that has been going on steadily since the creation of mathematics. We have arithmetic tables going back four or five millennia. We have "state of the art" summaries of mathematics prepared in classical antiquity (Pappus, *Collections*, c. 300). We have mechanical devices such as abaci, quipu, calculating tables and machines that are equally ancient.

Can one expect revolutions in mathematics? Certainly, but only rarely. If one defines a mathematical revolution or bombshell as a development that seriously alters the philosophy of mathematics, I can think of only four bombshells—a rather stingy assessment, I admit—in the past four centuries: Isaac Newton's *Principia* (1687), János Bolyai–Nikolai Lobachevsky's non-Euclidean geometry (1832–1840), Georg Cantor's set theory (1874), and Kurt Gödel's incompleteness theorem (1930). Figuring that the arrival of a bombshell takes perhaps a hundred years, the next one should be due around the year 2030. But contemporary mathematicians like to claim bombshells for their generation. Thus Peter Lax suggested that "Milnor's differential topology, which has altered our notion of how space hangs together," should be added to my list. David Mumford points out that during the 20th century there was a gradual and revolutionary shift toward an all-embracing abstract structuralism, associated with the German school of abstract algebra. and moving toward the work of the École Bourbaki and of Henri Cartan. “It was part of the larger cultural movement that brought us cubism and abstract art.” This parallelism between art and mathematics has not yet been adequately critiqued.

What will the next bombshell consist of? I don’t think that the solution of any of the current crop of unsolved mathematical problems, important as they may be, will lead to the overturn of the philosophical notions now held strongly. The total mechanization of mathematics, however—if it ever gets close to that—would be a change at the meta-level, and should rumble the philosophers quite a bit.

As Caroline Dunmore wrote, “Mathematics is conservative on the object-level and revolutionary on the meta-level.” I interpret this to mean that while the objects of mathematics, numbers, functions, geometrical figures, etc., largely retain their original personas, the formal superstructures in which they are embedded can change drastically (see, e.g., Saunders Mac Lane, cited below).

***

The promised future plethora of mathematics and mathematizations and the increasing approval and support by society leads inevitably to the question of whether mathematics is, as some claim, ethically neutral. I answer that mathematics is ethically ambiguous. Søren Kierkegaard’s famous *Either/Or* described the conflict between the aesthetic and the ethical. Kierkegaard opted for the ethical. Niels Bohr wrote an open letter to the United Nations (July 7, 1950) in which he argued for free scientific discourse as a means of promoting *détente*. Was this position naïve? Was it irrelevant? Or was it prophetic in that, e.g., the Internet has altered the attitudes and interactions of populations and hence of international relations?

The razor’s edge of ambiguity on which mathematics is now poised is well illustrated by the career of Lewis Fry Richardson (1881–1953), an applied mathematician, physical scientist, inventor, and sociometrist. Richardson was also a Quaker pacifist who is considered to be the father of modern mathematical war-gaming. Richardson believed that modeling warfare would lead to a diminution of antagonisms. War-gaming is now a technique employed by military strategists.

Language is a symbolic system that has raised us from the level of Caliban brutes and has accordingly transformed our lives and the same may be asserted of mathematics. It is a language that has transformed our lives for good and for bad. It has been the handmaiden not only of miraculous technology but also of new and unprecedented dimensions of human cruelty. While the ethical issues in sciences are widely recognized, the ethical issues in mathematical thinking are not adequately recognized, either within the mathematical research community or in mathematical education. So what has been our profit from it? Surely mathematics has not been a zero-sum game. Nonetheless, we must remain constantly aware of the painful truth that mathematics has the Promethean power both of life enhancement and of returning us to the condition of brutes.

Further Reading

V. Arnold, et al., eds. *Mathematics: Frontiers and Perspectives*. American Mathematical Society, 1999.

Michael Beeson. "The Mechanization of Mathematics." In *Alan Turing: Life and Legacy of a Great Thinker*, C. Teuscher, ed. Springer-Verlag, 2003. Also available from the Internet (http://www.cs.sjsu.edu/faculty/beeson/Papers/turing2.pdf).

Bernhelm Booss-Bavnbek and Jens Høyrup, eds. *Mathematics and War*. Birkhäuser, 2003.

Philip J. Davis. "War, Weather, and Mathematics." *SIAM News*, Vol. 29, No. 9, November 1996.

Philip J. Davis. "Biting the Bullet: Mathematics and War." *SIAM News*, Vol. 37, No. 2, March 2004.

Philip J. Davis. "The Power of Numerics." *SIAM News*, October, 2004.

Philip J. Davis. "Running Counter to Inert Crystallized Opinion." *SIAM News*, Vol. 37, December 2004.

Philip J. Davis. *The Prospects for Mathematics in a Multi-media Civilization.* Urania talk presented at the International Congress of Mathematicians, Berlin, 1998. Also in *Alles Mathematik*, Martin Aigner and Ehrhard Behrends, eds. Vieweg, 2000.

Caroline Dunmore. "Meta-Level Revolutions in Mathematics." In *Revolutions in Mathematics*, David Gillies, ed. Oxford University Press, 1992.

Björn Engquist and Wilfried Schmid, eds. *Mathematics Unlimited—2001 and Beyond*. Springer-Verlag, 2001.

Jeremy Gray. *János Bolyai, Non-Euclidean Geometry and the Nature of Space*. MIT Press, 2004.

Richard W. Hamming. "The Unreasonable Effectiveness of Mathematics." *American Mathematical Monthly*, Vol. 87, No. 2, February 1980.

Gerald Holton. "On the Psychology of Scientists, and Their Social Concern*s*." In *The Scientific Imagination: Case Studies.* Cambridge University Press, 1978.

Peter Lax. Letter to editor in *SIAM News*, Vol. 38, No. 1, Jan/Feb, 2005.

Saunders Mac Lane. *Mathematics, Form and Function*. Springer-Verlag, 1986.

David Mumford, e-communication.

Laurent Schwartz. *Laurent Schwartz: A Mathematician Grappling with his Century.* Birkhäuser, 2001.

Philip J. Davis is professor emeritus of Applied Mathematics at Brown University. His writings include numerous technical papers and books on complex variable theory, approximation theory, and numerical analysis, as well as on the philosophy of mathematics. His book ***The Mathematical Experience***, co-authored with Reuben Hersh, won an American Book Award. He has for many years been a columnist for the ***SIAM News***, a publication of the Society for Industrial and Applied Mathematics.

MATHEMATICS *and* COMMON SENSE

A Case of Creative Tension

Philip J. Davis

"Phil Davis is one of a very small group of mathematicians who are interested and able to step outside the community and take a hard look at what mathematics really 'is'. Its uses, misuses, customs, relations with the so-called 'real' world, psychology and deep nature are all grist for his voracious mill."
—David Mumford, Brown University

"It is risky to trust common sense—especially when dealing with mathematics. Convincingly Phil Davis demonstrates this, first by answering a few harmless-sounding 'frequently asked questions' about his discipline and then by skillfully seducing the reader into the curious kernel of the most fascinating of all sciences: Mathematics (sometimes written with a capital M, sometimes with a lowercase m; sometimes pure, sometimes applied; sometimes familiar, sometimes strange, even to experts)."
—Rudolf Taschner (Vienna), Austrian Scientist of the Year 2004

"Drawing on a long experience in mathematics as a researcher, teacher and popularizer, Davis reflects upon the subject from many points of view: not only its development and instruction, but also its overly modest status in society in general."
—Ivor Grattan-Guinness

A K Peters, Ltd.

ISBN 1-56881-270-1

9 781568 812700